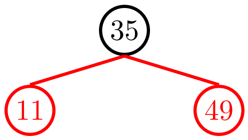
1 On Wednesdays We Wear Red & Black

1.1 Here is a refresher on inserting into a red-black tree:

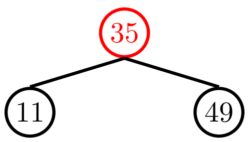
```java
public void put(Key key, Value val) {
    root = put(root, key, val);
    root.color = BLACK;
}

private Node put(Node h, Key key, Value val) {
    if (h == null){
        return new Node(key, val, RED, 1);
    }
    int cmp = key.compareTo(h.key);
    if  (cmp < 0) {
        h.left  = put(h.left,  key, val);
    } else if (cmp > 0) {
        h.right = put(h.right, key, val);
    } else {
         h.val = val;
    }
    if (isRed(h.right) && !isRed(h.left)) {
         h = rotateLeft(h);
    }
    if (isRed(h.left)  &&  isRed(h.left.left)) {
        h = rotateRight(h);
    }
    if (isRed(h.left)  &&  isRed(h.right)) {
        flipColors(h);
    }
    h.size = size(h.left) + size(h.right) + 1;

    return h;
}
```

Now draw out the left leaning red black tree resulting from inserting the following- 35, 11, 49, 9, 7, 51 and 50.

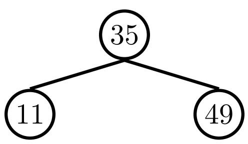
Insert 35, 11 and 49



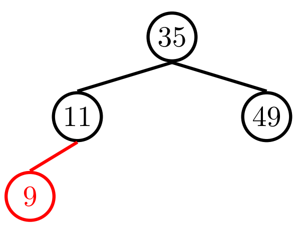
Flip Colors



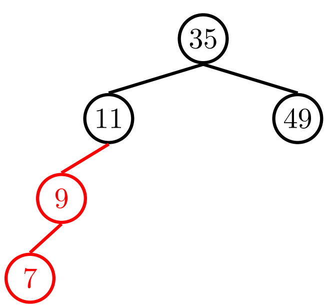
Color Root Black



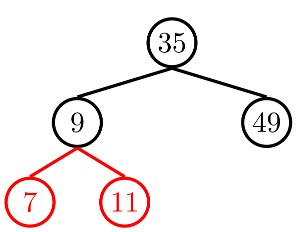
Insert 9



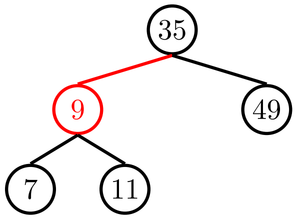
Insert 7



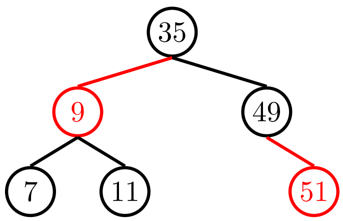
Rotate Right



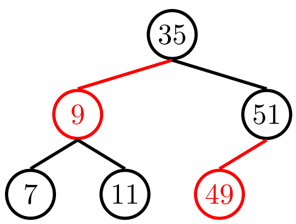
Flip Colors



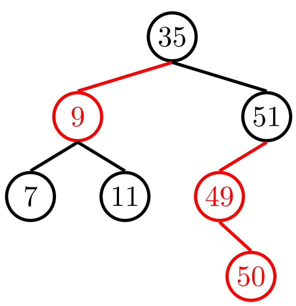
Insert 51



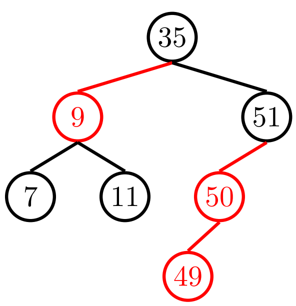
Rotate Left



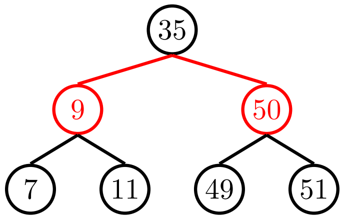
Insert 50



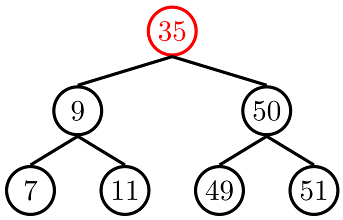
Rotate Left



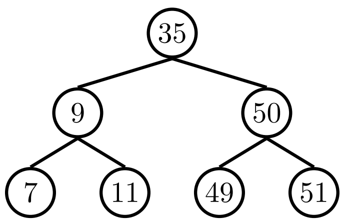
Rotate Right



Flip Colors



Color Root Black



2 Asymptotics

2.1 Please give lower and upper bounds for the overall runtime of these functions.

```
void g(double N) {
    if (N<=1)
        return;
    if (N%2 == 0) {
        for(double i = N/2; i >= N/4; i = i/2) {
                func(i) // linear with respect to i
        }
    }
    g(N/2)
}
```

In this function, the runtime depends on whether the starting value of N is odd or a power of 2. If N is odd then the condition of (N%2 == 0) is never satisfied. Since N is a double, there is no case that dividing by 2 will yield another even number. As such, on each function call, the function does a constant amount of work. The function is called log(N) times since N is divided by two on each recursive call. This yields a runtime of Omega(logN).

The condition of (N%2 == 0) is always satisfied on each recursive call if N is a power of 2. On the first call, when the argument is N, the function does 3N/4 work. On the next recursive call, when the argument is N/2, the function does 3N/8 work. On the next call, when the argument is N/4, the function does 3N/16 work. We see a pattern forming where the amount of work being done is roughly $3N/2^{(k+1)}$ where k denotes the level of the recursive tree we are on. When summing all the runtimes across recursive calls together, the runtime is

$$3N/4+3N/8+3N/16+\ldots+3N/2^{logN} = 3N*(1/4+1/8+1/16+\ldots+1/2^{logN}) = N$$

So in the worst case the runtime is O(N)

2.2 Under what conditions are the best case runtime achieved. Under what conditions are the worst case runtime achieved

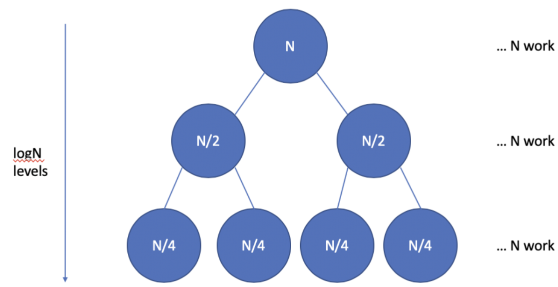
```
// x has nonnegative integers and the size is M
void t(int[] x; int N) {
        boolean flag = true;
    while (flag) {
            flag = false;
            for (int i = 0; i < x.length; i++) {
                    if(x[i] < N) {
                            x[i] += 1;
                flag = true;
            }
        }
    }
}
```

The best case runtime is achieved if all of the elements have values greater than N. In this case, the program will loop once over all the elements in the array and the lower bound overall runtime is $\Omega(M)$

The worst case runtime is achieved if at least one element has a value of 0. This is because on each loop over the entire array, elements are only incremented by 1. This means that if a single element has a value of zero, the program will need to loop over the array N times to increase the element in question to N. This means that the overall worst case runtime is bounded by O(MN). Note the this worst case runtime is met if there is only a single 0 in the array and if all the elements in the array are 0.

2.3 Give a tight bound on the worst case runtime of the following function

```
int f(int N) {
        if (N == 1)
                return 1;
        int y = 0;
        for (int x = 0; x < N; x += 1) {
                y += 1;
        }
        return f(N/2) + f(N/2) + y;
}
```

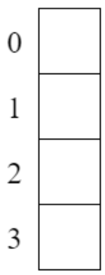


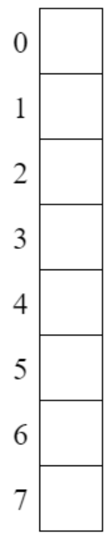
The branching factor at each node is two because on each function call, 2 recursive calls are made. Due to the for loop, each function does a linear amount of work with respect to its argument. This is represented in the tree through the value in each of the nodes. There are logN levels since N is divided by two on each recursive call. Since there are logN levels and each level does a total of N work, the runtime is $\Theta(NlogN)$

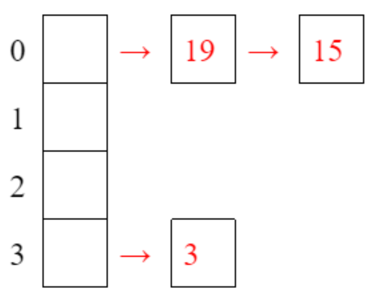
3 Hashing

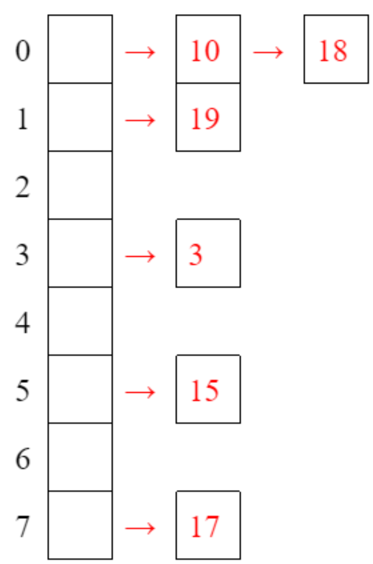
3.1 Consider a hash table with external chaining, using the hash function $h(x) = x \mod 10$. The table resizes when the load factor exceeds 0.75. Draw out the hash table as we insert the following values:

3, 19, 15, 10, 17, 18









3.2 Here's a class for a hashtable that takes key, value pairs and stores them in a hashtable with external chaining. The keys in this problem are all strings, and the hashcode for each string is its length. Implement the hash function `hashfunc(String key)` and complete the method `insert` that inserts new key value pairs into the hashtable. You will be implementing the part of `insert` that handles resizing. It should be done recursively.

```
// A node of the external chains
class HashNode<String, V> {
    String key;
    V value;
    HashNode<String, V> next;

    public HashNode(String key, V value) {
        this.key = key;
        this.value = value;
    }
}

// class for the actual table
class HashTable<String, V> {
    private HashNode<String,V>[] hashtable;
    private int numBuckets;
    private int numElements;
    private int maxLoad = 0.75;

    public HashTable() {
        numBuckets = 10;
        numElements = 0;
        hashtable = new HashNode<String,V>[numBuckets];

        // Create empty chains
        for (int i = 0; i < numBuckets; i++)
            hashtable[i] = null;
    }

    // Given a key, returns the hashcode for that key
    private int hashfunc(String key) {
        return __________________;
    }

    public void insert(String key, V value) {
        int index = hashfunc(key);
        HashNode<String, V> head = hashtable[index];

        //check if key is already in table
        while (head != null) {
```

```
            if (head.key.equals(key)) {
                head.value = value;
                return;
            }
            head = head.next;
        }

        // resize
        if (_______________________) {
            numBuckets = ___________________;
            HashNode<String,V>[] temp = hashtable;
            hashtable = new HashNode<String,V>[numBuckets];
            numElements = _____________;

            for (int i = 0; i < numBuckets; i++) {
                ________________________________;
        }

            for (HashNode<String, V> headNode : __________) {
                while (_________________________) {
                    ________________________________________;
                    ________________________________________;
                }
            }
        }

        numElements++;
        HashNode<String, V> newNode = new HashNode<String, V>(key, value);
        newNode.next = head;
        hashtable[index] = newNode;
    }
}
```

```
private int hashfunc(K key) {
    return key.length();
}

// resize
if ((numElements+1)/numBuckets > maxLoad) {
    numBuckets *= 2;
    HashNode<K,V>[] temp = hashtable;
    hashtable = new HashNode<K,V>[numBuckets];
```

```
    numElements = 0;

    for (int i = 0; i < numBuckets; i++)
        hashtable.add(null);

    for (HashNode<K, V> headNode : temp) {
        while (headNode != null) {
            insert(headNode.key, headNode.value);
            headNode = headNode.next;
        }
    }
}
```

4 Heaps

4.1 You have been hired by Alan to help design a priority queue implementation for *Kelp*, the new seafood review startup, ordered on the timestamp of each `Review`.

Describe a data structure that supports the following operations.

- `insert(Review r)` a `Review` in $O(\log N)$.
- `edit(int id, String body)` any one `Review` in $\Theta(1)$.
- `sixtyOne()`: return the sixty-first latest `Review` in $\Theta(1)$.
- `pollSixtyOne()`: remove and return the sixty-first latest `Review` in $O(\log N)$.

Maintain a max-heap called `firstSixtyOne` with 61 `Reviews`, a min-heap called `olderReviews` with all the rest, and a `HashMap` mapping any given integer `id` to its corresponding `Review`.

5 KD-Trees

5.1 1. The root node divides the space into...

   - 2 quadrants, positive and negative x
   - 2 quadrants, positive and negative y
   - 4 quadrants
   - None of the above

2. Suppose you have just one node (65, 12). You want to insert (1, 12). Where should this node be placed in the tree?

   - Left subtree
   - Right subtreee

3. Is the node with value (1, 12) x-aligned or y-aligned?

   - X-aligned
   - Y-aligned

4. Suppose after adding (1, 12), you decide to add (3, 15). Should this node be placed in the left or right subtree of (1, 12)?

   - Left
   - Right

5. When we want to find the minimum element of a specific dimension D, we like to go through the tree and ask at each dimension if the dimension we are currently standing in and D are the same. If true, then the minimum element is in...

   - The current node
   - The right subtree
   - The left subtree
   - An above node

6. If false, then the minimum element is in...

   - The current node
   - The right subtree
   - The left subtree
   - An above node

7. The average and worst runtime, respectively, for searching is...

- O(N), O(N)
- O(logN), O(logN)
- O(N), O(N*N)
- O(logN), O(N)

8. The average and worst runtime, respectively, for searching is. . .

    - O(N), O(N)
    - O(logN), O(logN)
    - O(N), O(N*N)
    - O(logN), O(N)

9. The space complexity is...

    - O(N)
    - O(logN)
    - O(N*N)
    - O(NlogN)

KD Trees

The root node divides the space into…

a. **2 quadrants, positive and negative x**
b. 2 quadrants, positive and negative y
c. 4 quadrants
d. None of the above

Suppose you have just one node (65, 12). You want to insert (1, 12). Where should this node be placed in the tree?

a. Left subtree
b. Right subtree

Is the node with value (1, 12) x-aligned or y-aligned? _____ **Y-aligned**

Suppose after adding (1, 12), you decide to add (3, 15). Should this node be placed in the left or right subtree of (1, 12)? _____ **Right**

When we want to find the minimum element of a specific dimension *D*, we like to go through the tree and ask at each dimension if the dimension we are currently standing in and *D* are the same. If true, then the minimum element is in…

a. The current node
b. The right subtree
c. **The left subtree**
d. An above node

If false, then the minimum element is in…

a. The current node
b. The right subtree
c. The left subtree
d. **Any of the above**

The average and worst runtime, respectively, for searching is…

a. O(N), O(N)
b. O(logN), O(logN)
c. O(N), O(N*N)
d. **O(logN), O(N)**

The average and worst runtime, respectively, for insertion and deletion is…

a. O(N), O(N)
b. **O(logN), O(N)**
c. O(logN), O(logN)
d. O(N), O(N*N)

The space complexity is ____. **O(N)**